Prince Sultan University

Math 113
Major Exam 3
Second Semester, Term 152
Thursday, April 28, 2016

Time Allowed: 90 minutes

Student Name:________________ ____________________ ______________________

Student ID #: ___________________

Serial Class #:_____ Section #:

Instructor's Name:

Important Instructions:

1. You may use a scientific calculator that does not have programming or graphing capabilities.
2. You may NOT borrow a calculator from anyone.
3. You may NOT use notes or any textbook.
4. Talking during the examination is NOT allowed.
5. Your exam will be taken immediately if your mobile phone is seen or heard.
6. Looking around or making an attempt to cheat will result in your exam being cancelled.
7. This examination has 9 problems, some with several parts. Make sure your paper has all these problems.

Problems	Max marks	Student's marks
Q#1+Q#2	17	
Q#3	18	
Q#4+Q#5	21	
Q#6	24	
Total	80	

Q#1. [7 Marks] Find the sum of $\sum_{n=1}^{+\infty}[\tan^{-1}(n+1)-\tan^{-1}n]$

$$a_n = \tan^{-1}(n+1) - \tan^{-1}(n)$$

$$S_1 = a_1 = \tan^{-1}2 - \tan^{-1}1$$

$$S_2 = a_1 + a_2 = (\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) = \tan^{-1}3 - \tan^{-1}1$$

$$S_3 = a_1 + a_2 + a_3 = (\tan^{-1}3 - \tan^{-1}1) + \tan^{-1}4 - \tan^{-1}3 = \tan^{-1}4 - \tan^{-1}1$$

So $S_n = \tan^{-1}(n+1) - \tan^{-1}1$

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} \tan^{-1}(n+1) - \tan^{-1}1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}. \text{ So } \sum_{n=1}^{\infty} \tan^{-1}(n+1) - \tan^{-1}n = \frac{\pi}{4}$$

Q#2[4+4+2 Marks] Find the sum of the following series:

1. $\sum_{n=1}^{+\infty}\frac{1}{n(n+1)}$

$$a_n = \frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1} = \frac{A(n+1) + Bn}{n(n+1)}$$

So $1 = A(n+1) + Bn$. If $n=0$, then $A=1$
If $n=-1$, then $B=-1$

So $a_n = \frac{1}{n} - \frac{1}{n+1}$

$S_1 = 1 - \frac{1}{2}$, $S_2 = (1-\frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) = 1 - \frac{1}{3}$. So $S_n = 1 - \frac{1}{n+1}$.

$$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty}\left(1 - \frac{1}{n+1}\right) = 1.$$

So $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$

2. $\sum_{n=1}^{+\infty}\frac{3}{3^n}$.

$$= \sum_{n=1}^{\infty} 3\left(\frac{1}{3}\right)^n = \frac{3(\frac{1}{3})}{1 - \frac{1}{3}} = \frac{1}{\frac{2}{3}} = \frac{3}{2}$$

3. $\sum_{n=1}^{+\infty}\left[\frac{3}{n(n+1)} + \frac{6}{3^n}\right]$ $= 3\sum_{n=1}^{+\infty}\frac{1}{n(n+1)} + 2\sum_{n=1}^{\infty}\frac{3}{3^n}$

$$= 3 + 2\left(\frac{3}{2}\right) = 3 + 3 = 6$$

Q#3.[6 Marks each] Test the following series for convergence or divergence:[In order to get the full mark you have to show all your work and to write the name of the test you want to use]

1. $1+\frac{1}{2\sqrt{2}}+\frac{1}{3\sqrt{3}}+\frac{1}{5\sqrt{5}}+..........$ $= \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$

p_series $p = \frac{3}{2} > 1$

So the series converges.

2. $\sum_{k=2}^{\infty} \frac{e^k}{(2+\frac{1}{k})^k}$

$a_k = \frac{e^k}{(2+\frac{1}{k})^k} \Rightarrow (a_k)^{1/k} = \frac{e}{2+\frac{1}{k}}$

$L = \lim_{k\to+\infty} (a_k)^{1/k} = \lim_{k\to+\infty} \frac{e}{2+\frac{1}{k}} = \frac{e}{2} > 1$

By root test, $\sum_{k=2}^{+\infty} \frac{e^k}{(2+\frac{1}{k})^k}$ diverges.

3. $\sum_{n=1}^{\infty} \frac{\sqrt{2n(n-7)(n-9)}}{3n^2-4n+9}$

$a_n = \frac{\sqrt{2n(n-7)(n-9)}}{3n^2-4n+9}$. Take $b_n = \frac{\sqrt{2n^3}}{3n^2} = \frac{\sqrt{2}\, n^{\frac{3}{2}}}{3n^2}$

$= \frac{\sqrt{2}}{3 n^{\frac{1}{2}}}$.

$\frac{a_n}{b_n} = \frac{\sqrt{2n(n-7)(n-9)}}{3n^2-4n+9} \cdot \frac{3n^{\frac{1}{2}}}{\sqrt{2}}$. Note that $\lim_{n\to+\infty} \frac{a_n}{b_n} = 1$

So Both series converge or diverge

by limit comparsion test

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{3n^{\frac{1}{2}}}$ diverges (p-series $p = \frac{1}{2} < 1$),

by limit comparsion test $\sum_{n=1}^{+\infty} \frac{\sqrt{2n(n-7)(n-9)}}{3n^2-4n+9}$ diverges

Q#4.[6 Marks each] Test the following series for convergence or divergence:[In order to get the full mark you have to show all your work and to write the name of the test you want to use]

1. $\sum_{n=4}^{+\infty} \frac{\ln n}{n}$

Take $f(x) = \frac{\ln x}{x}$.
Note that f is cts, decreasing and positive on $[4, +\infty)$. Look to

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$$\int_4^{+\infty} \frac{\ln x}{x}\,dx = \lim_{t\to+\infty} \int_4^t \frac{\ln x}{x}\,dx = \lim_{t\to+\infty} \left.\frac{(\ln x)^2}{2}\right]_4^t = \lim_{t\to+\infty} \frac{(\ln t)^2}{2} - \frac{(\ln 4)^2}{2} = +\infty$$

So by $\boxed{\text{integral test}}$, the series diverges.

2 2

OR $\frac{\ln n}{n} \geq \frac{1}{n}$, $n \geq 1$ & $\sum \frac{1}{n}$ diverges $\Rightarrow \sum \frac{\ln n}{n}$ div.

2. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$

$\boxed{\text{Alternating series}}$ 1

$a_n = \frac{1}{\ln n}$

(i) $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{1}{\ln n} = 0$

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(ii) Take $f(x) = \frac{1}{\ln x}$, $f'(x) = \frac{-\frac{1}{x}}{(\ln x)^2} = \frac{-1}{x(\ln x)^2}$

$f'(x) < 0$ on $[2, +\infty)$. So (a_n) is decreasing. So the series converges. 1

Q#5[3 Marks each] Test the following sequences for convergence or divergence. [Justify your answer]

1. $\left(\frac{n}{n^2-1}\right)$

$a_n = \frac{n}{n^2-1}$

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} \frac{n}{n^2-1} = \lim_{n\to+\infty} \frac{1}{2n} = 0$$

So $\left(\frac{n}{n^2-1}\right)$ converges to zero.

2. $\left(\frac{\sin(2n)}{n}\right)$

$-1 \leq \sin(2n) \leq 1 \Rightarrow \frac{-1}{n} \leq \frac{\sin(2n)}{n} \leq \frac{1}{n}$

$\lim_{n\to+\infty} \frac{-1}{n} = \lim_{n\to+\infty} \frac{1}{n} = 0$. So by squeezing thrm

$\lim_{n\to+\infty} \frac{\sin(2n)}{n} = 0$. So $\left(\frac{\sin(2n)}{n}\right)$ converges to 0.

3. $(n - \ln n)$

$$n - \ln n = \ln(e^n) - \ln n = \ln\left(\frac{e^n}{n}\right)$$

$$\lim_{n\to+\infty} \frac{e^n}{n} = \lim_{n\to+\infty} \frac{e^n}{1} = +\infty.$$

Since $\ln x$ is an increasing function and $\frac{e^n}{n} \to +\infty$, we conclude that $(n - \ln n)$ diverges.

OR $n\left(1 - \frac{\ln n}{n}\right) \xrightarrow{LR} \infty \cdot 1 = \infty$

Q#6[6 Marks each] Classify the following series to convergence, absolutely convergence, and conditionally convergence:

1. $\sum_{n=1}^{+\infty} \frac{3n^2+7}{4n^2+5n+3}$

$a_n = \frac{3n^2+7}{4n^2+5n+3}$. Then $\lim_{n\to+\infty} a_n = \frac{3}{4} \neq 0$. So by divergent theorem $\sum_{n=1}^{\infty} \frac{3n^2+7}{4n^2+5n+3}$ diverges. (and hence ~~...~~ conditionally divergent.)

Note that $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} a_n$, so the series is absolutely divergent.

2. $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$

(a) Take $a_n = \frac{1}{\sqrt{n}}$ (i) $\lim_{n\to+\infty} \frac{1}{\sqrt{n}} = 0$ (ii) $(a_n) = (1, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{3}}, \dots)$ is ~~decreasing~~ decreasing. So by Alternating series test, the series Converges.

(b) look to $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges p-series $p = \frac{1}{2} < 1$

So the series diverges absolutely

(c) Since the series converges but diverges absolutely. So conditionally convergent

3. $\sum_{n=1}^{\infty} (-1)^{n+5} \frac{(3)^n}{(2n+1)!}$

$a_n = (-1)^{n+1} \frac{4^n}{(2n+1)!} \Rightarrow |a_n| = \frac{3^n}{(2n+1)!}$, $|a_{n+1}| = \frac{3^{n+1}}{(2n+3)!}$

$\frac{|a_{n+1}|}{|a_n|} = \frac{3^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{4^n} = \frac{3^n \cdot 3 \cdot (2n+1)!}{(2n+3)(2n+2)(2n+1)! \cdot 4^n} = \frac{3}{(2n+3)(2n+2)}$

$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \frac{3}{4} < 1$. (The series is absolutely convergent and hence convergent)

So conditionally divergent

4. $\sum_{n=1}^{+\infty} \frac{\sin(4n)}{1+5^n}$

$|\sin(4n)| \leq 1 \Rightarrow \frac{|\sin(4n)|}{1+5^n} < \frac{1}{1+5^n} \leq \frac{1}{5^n}$

Since $\sum_{n=1}^{\infty} \frac{1}{5^n}$ converges (geometric series),

we deduce that $\sum_{n=1}^{\infty} \frac{\sin(4n)}{1+5^n}$ is absolutely convergent and hence converges.

So not Conditionally convergent.