Prince Sultan University

Deanship of Educational Services

Department of Mathematics
and General Sciences

جامعة الأمير سلطان
PRINCE SULTAN
UNIVERSITY
1998

Solutions

COURSE DETAILS:

Calculus II	MATH 113	MAJOR EXAM II
Semester:	Spring Semester --Term 172	
Date:	Wednesday April 11, 2018	
Time Allowed:	90 minutes	

STUDENT DETAILS:

Student Name:	
Student ID Number:	
Section:	
Instructor's Name:	

INSTRUCTIONS:

- You may use a scientific calculator that does not have programming or graphing capabilities. NO borrowing calculators.
- NO talking or looking around during the examination.
- NO mobile phones. If your mobile is seen or heard, your exam will be taken immediately.
- Show all your work and be organized.
- You may use the back of the pages for extra space, but be sure to indicate that on the page with the problem.

GRADING:

	Page 1	Page 2	Page 3	Page 4	Total	Total
Questions	1,2,3	4	5,6,7	8	8	
Marks	16	16	16	12	60	20

Q#1 [5 Marks] Find the average value of the function $f(x)=\ln\sqrt[4]{x}$ from $x=1$ to $x=4$. <u>Give the exact answer</u>.

$$f_{ave} = \frac{1}{4-1}\int_1^4 \ln\sqrt[4]{x}\,dx \quad (1) = \frac{1}{3}\cdot\frac{1}{4}\int_1^4 \ln x\,dx \quad (1)$$

$u=\ln x \qquad dv=dx$

$du=\frac{1}{x}dx \qquad v=x \quad (1)$

$$= \frac{1}{12}\left[x\ln x - x\right]_1^4 \quad (1)$$

$$= \frac{1}{12}\left[4\ln 4 - 4 - (\ln 1 - 1)\right]$$

$$= \frac{1}{12}(4\ln 4 - 3) = \frac{-3+4\ln 4}{12} \quad (1)$$

Q#2 [4 Marks] Use the comparison principle to test the convergence of $\int_1^\infty \frac{\sin x+1}{x^3}dx$.

$\sin x \le 1 \Rightarrow \sin x + 1 \le 2 \quad (1)$

$\Rightarrow \frac{\sin x+1}{x^3} \le \frac{2}{x^3} \quad (2)$ since $\int_1^\infty \frac{2}{x^3}dx$ is convergent because $p>1$,

then by Compar. the. $\int_1^\infty \frac{\sin x+1}{x^3}dx$ converges (1)

Q#3 [7 Marks] Evaluate the integral: $\int_0^\infty \frac{dx}{x^2+3x+2}$ if it is convergent.

$$\int_0^\infty \frac{dx}{x^2+3x+2} = \lim_{t\to\infty}\int_0^t \frac{dx}{(x+2)(x+1)} \quad (1) = \lim_{t\to\infty}\int_0^t \left(\frac{-1}{x+2}+\frac{1}{x+1}\right)dx \quad (2)$$

$$= \lim_{t\to\infty}\left(-\ln|x+2| + \ln|x+1|\right)_0^t \quad (1)$$

$$= \lim_{t\to\infty}\left[\left(-\ln|t+2| + \ln|t+1|\right) - \left(-\ln 2 + \ln 1\right)\right]$$

$$= \lim_{t\to\infty}\left(\ln\frac{|t+1|}{|t+2|} + \ln 2\right) \quad (1)$$

$$= \ln \lim_{x\to\infty}\frac{x+1}{x+2} + \ln 2 \stackrel{(H)}{=} \ln 1 + \ln 2 \quad (1) = \ln 2 \quad (1)$$

It converges to $\ln 2$

Q#4 **[5+6+5 Marks]** Evaluate each integral:

1. $\int \sqrt{1+\sqrt{x}}\, dx$

Let $u = 1+\sqrt{x} \Rightarrow u-1 = \sqrt{x} \Rightarrow (u-1)^2 = x$ (2)

$2(u-1)\,du = dx$

$= 2\int \sqrt{u}\,(u-1)\,du$ (1)

$= 2\int (u^{3/2} - u^{1/2})\,du = \frac{4}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C$ (1)

$= \frac{4}{5}(1+\sqrt{x})^{5/2} - \frac{4}{3}(1+\sqrt{x})^{3/2} + C$ (1)

2. $\int \frac{\sqrt{x^2-1}}{x^4}\, dx$

Let $x = \sec\theta \Rightarrow dx = \sec\theta\tan\theta\, d\theta$ (1)

$\sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta} = \tan\theta$

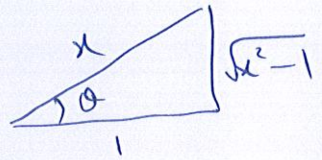
$= \int \frac{\tan\theta \sec\theta \tan\theta\, d\theta}{\sec^4\theta}$ (1)

let $u = \sin\theta$

$du = \cos\theta\, d\theta$

$= \int \tan^2\theta \cos^3\theta\, d\theta = \int \sin^2\theta \cos\theta\, d\theta$ (1)

$= \int u^2\, du = \frac{u^3}{3} + C = \frac{\sin^3\theta}{3} + C$ (2)



$= \frac{(\sqrt{x^2-1})^3}{3x^3} + C$ (1)

3. $\int \tan^4 x \sec^4 x\, dx$

let $u = \tan\theta \Rightarrow du = \sec^2\theta\, d\theta$ (1)

$\int \tan^4\theta\,(\tan^2\theta + 1)\sec^2\theta\, d\theta$ (1)

$= \int (u^6 + u^4)\,du$ (1) $= \frac{u^7}{7} + \frac{u^5}{5} + C$ (1)

$= \frac{\tan^7\theta}{7} + \frac{\tan^5\theta}{5} + C$ (1)

Q#5 [4 Marks] Obtain the arc length of the portion of the curve $y = x\sqrt{x}$ from $(1,1)$ to $(4,8)$.

$$y = x^{\frac{3}{2}} \Rightarrow y' = \frac{3}{2}x^{\frac{1}{2}} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\left(\frac{3}{2}x^{\frac{1}{2}}\right)^2 = 1+\frac{9}{4}x \quad 1$$

$$L = \int_1^4 \sqrt{1+\frac{9}{4}x}\, dx \quad 1$$

$$= \frac{2}{3}\left(1+\frac{9}{4}x\right)^{\frac{3}{2}} \cdot \frac{4}{9}\Big]_1^4 = \frac{8}{27}\left(10^{\frac{3}{2}} - \left(\frac{13}{4}\right)^{\frac{3}{2}}\right) \simeq 7.6$$

1 1

Q#6 [3+4 Marks] Setup the integral for surface area when the curve $y = \ln x,\ 1 \le x \le e$ is rotated

(a) about the x-axis. [Do not evaluate the integral]

$$\frac{dy}{dx} = \frac{1}{x} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{x^2} \quad 1$$

$$S = \int_1^e 2\pi \ln x \sqrt{1+\frac{1}{x^2}}\, dx \quad 2$$

$$x = e^y \Rightarrow \frac{dx}{dy} = e^y \quad 1$$

or

$$1+\left(\frac{dx}{dy}\right)^2 + 1 + e^{2y} \quad 1$$

or $$S = \int_0^1 2\pi y \sqrt{1+e^{2y}}\, dy$$

$$1 \le x \le e \Rightarrow 0 \le y \le 1$$

(b) about the y-axis. [Do not evaluate the integral]

$$S = \int_1^e 2\pi x \sqrt{1+\frac{1}{x^2}}\, dx \quad 2$$

or $$S = \int_0^1 2\pi e^y \sqrt{1+e^{2y}}\, dy$$

Q#7 [5 Marks] Determine whether the integral $\int_0^3 \frac{1}{(2x-5)^4}\,dx$ is convergent or divergent. Give the value of the integral if it converges.

$$= \underbrace{\int_0^{5/2} \frac{1}{(2x-5)^4}\, dx}_{I_1} + \underbrace{\int_{5/2}^3 \frac{1}{(2x-5)^4}\, dx}_{I_2} \quad 1$$

$$I_1 = \lim_{t \to \frac{5}{2}^-} \int_0^t \frac{1}{(2x-5)^4}\, dx \quad 1 \quad = \lim_{t \to \frac{5}{2}^-} \frac{(2x-5)^{-3}}{-6}\Big|_0^t \quad 1$$

$$= \lim_{t \to \frac{5}{2}^-} \left(\frac{(2t-5)^{-3}}{-6} - \frac{(-5)^{-3}}{-6}\right) \quad 1$$

$$= \lim_{t \to \frac{5}{2}^-} \left(\frac{1}{-6(2t-5)^3} - \frac{1}{750}\right) = \infty \quad 1$$

$\Rightarrow$ integral diverges

Q#8 [4 Marks each] Determine whether each sequence converges or diverges. If it converges, find the limit

(a) $\{a_n\} = \left\{\dfrac{5^n}{4+3^n}\right\}$

$$\lim_{n\to\infty} \frac{5^n}{4+3^n} = \lim_{n\to\infty} \frac{1}{\frac{4}{5^n} + \left(\frac{3}{5}\right)^n} = \infty \Rightarrow \text{div.}$$

2 2

or
$$\lim_{n\to\infty} \frac{5^n}{4+3^n} = \lim_{x\to\infty} \frac{5^x}{4+3^x} = \lim_{x\to\infty} \frac{5^x \ln 5}{3^x \ln 3} = \lim_{x\to\infty} \frac{\ln 5}{\ln 3}\left(\frac{5}{3}\right)^x$$
2

$$= \infty \Rightarrow \text{div.}$$
2

since $r = \frac{5}{3} > 1$

(b) $\{a_n\} = \left\{\dfrac{(\ln 2)^2}{-2}, \dfrac{(\ln 3)^2}{3}, \dfrac{(\ln 4)^2}{-4}, \ldots\right\}$

$$a_n = \frac{(-1)^n (\ln(n+1))^2}{n+1}$$
1

$$\lim_{n\to\infty} |a_n| = \lim_{x\to\infty} \frac{(\ln(x+1))^2}{x+1} \overset{H}{=} \lim_{x\to\infty} \frac{2\ln(x+1)\cdot\frac{1}{x+1}}{1}$$
1

$$= \lim_{x\to\infty} \frac{2\ln(x+1)}{x+1} \overset{H}{=} \lim_{x\to\infty} \frac{2}{x+1} = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$$
1

$\{a_n\}$ converges to 0 1

(c) $\{a_n\} = \left\{2e^{-\frac{1}{n}} + 3e^{-n}\right\}$

$$\lim_{n\to\infty}\left(2e^{-\frac{1}{n}} + 3e^{-n}\right)$$
2

$$= 2\cdot e^0 + 0 = 2$$

Conv. to 2 2