Prince Sultan University

Math 113
Major Exam 2
Second Semester, Term 152
Thursday, March 31, 2016

Time Allowed: 90 minutes

Student Name:______________ _________________ ____________________

Student ID #: _________________

Serial Class #:_____ Section #:

Instructor's Name:

Important Instructions:

1. You may use a scientific calculator that does not have programming or graphing capabilities.
2. You may NOT borrow a calculator from anyone.
3. You may NOT use notes or any textbook.
4. Talking during the examination is NOT allowed.
5. Your exam will be taken immediately if your mobile phone is seen or heard.
6. Looking around or making an attempt to cheat will result in your exam being cancelled.
7. This examination has 9 problems, some with several parts. Make sure your paper has all these problems.

Problems	Max marks	Student's marks
Q#1+Q#2	30	
Q#3	10	
Q#4+Q#5	14	
Q#6	8	
Q#7+Q#8+Q#9	18	
Total	80	

Q#1[14 Marks] Classify the following integrals to improper and not improper and write the reason:

	$\int_0^1 \ln x\,dx$	$\int_0^1 \frac{x}{x^2+2}dx$	$\int_{-2}^1 \frac{x}{x^3+8}dx$	$\int_0^{+\infty} \frac{x}{x^2+2}dx$	$\int_0^1 \frac{x+7}{x^2-2}dx$	$\int_0^{\frac{\pi}{2}} \tan x\,dx$	$\int_1^{10} (x^{10}+3)dx$
Improper	✓		✓	✓		✓	
Not improper		✓			✓		✓
The reason	$f(x)=\ln x$ is discts at 0	$f(x)=\frac{x}{x^2+2}$ is cts on $[0,1]$	$f(x)=\frac{x}{x^3+8}$ is discts at -2	$+\infty$ is one of the integral bound	$f(x)=\frac{x+7}{x^2-2}$ is cts on $[0,1]$	$f(x)=\tan x$ is discts at $x=\frac{\pi}{2}$	$f(x)=x^{10}+3$ is cts on $[1,10]$

10

Q#2. [16 Marks] Write the **best substitution** that can be used to evaluate the following integrals:

Recheck

The integral	The best substitution
$\int \frac{4}{\sqrt[3]{x}+4}dx$	$x=u^3$ or $u=\sqrt[3]{x}$
$\int \frac{2}{\sqrt[3]{x}+\sqrt[4]{x}}dx$	$x=u^{12}$ or $u=\sqrt[12]{x}$
$\int \frac{3}{\sqrt{x^2+2x+5}}dx$	$x+1=2\tan\theta$ $u=x+1$
$\int \sin^5 x\cos^6 x\,dx$	$u=\cos x$
$\int (\tan^2 x+\tan^4 x)\,dx$	$u=\tan x$
$\int x^3\sqrt{1-x^2}\,dx$	$x=\sin\theta$
$\int xe^{2\sqrt{x}}dx$	$u^2=x$ or ~~$x=\sqrt{u}$~~
$\int \frac{dx}{\sqrt{x^2-8x+12}}$	$x-4=2\sec\theta$

$\int \frac{3}{\sqrt{(x+1)^2+4}}dx$

2

$u=\sqrt{x}$, $u=2\sqrt{x}$

$x=u^2$, $4x=u^2$

ok ok

$u=\sqrt{1-x^2}$, $u=1-x^2$

$\int \frac{dx}{\sqrt{(x-4)^2-4}}$

$x-4=2\sec\theta$

Q#3[10 Marks] Evaluate $\int \frac{\sqrt{x}}{x^2-16}dx$

put $x = u^2 \Rightarrow dx = 2u\,du$

$$\int \frac{\sqrt{x}}{x^2-16}dx = \int \frac{u}{u^4-16}\cdot 2u\,du = \int \frac{2u^2}{u^4-16}du$$

$$\frac{2u^2}{u^4-16} = \frac{2u^2}{(u^2+4)(u^2-4)} = \frac{2u^2}{(u-2)(u+2)(u^2+4)}$$

$$= \frac{A}{u-2} + \frac{B}{u+2} + \frac{Cu+D}{u^2+4}$$

$$= \frac{A(u+2)(u^2+4) + B(u-2)(u^2+4) + (Cu+D)(u-2)(u+2)}{(u-2)(u+2)(u^2+4)}$$

$$2u^2 = A(u+2)(u^2+4) + B(u-2)(u^2+4) + (Cu+D)(u-2)(u+2)$$ 2

put $u=2 \Rightarrow 8 = 32A \Rightarrow A = \frac{1}{4}$

put $u=-2 \Rightarrow 8 = -32B \Rightarrow B = -\frac{1}{4}$

put $u=0 \Rightarrow 0 = 8A - 8B - 4D$

$0 = 2 - -2 - 4D \Rightarrow 0 = 4 - 4D \Rightarrow D = 1$

put $u=1 \Rightarrow 2 = 15A - 5B + (C+D)(-3)$

$\Rightarrow 2 = \frac{15}{4} + \frac{5}{4} - 3C - 3$

$\Rightarrow 2 = 5 - 3C - 3 \Rightarrow -3C = 0 \Rightarrow C = 0$

2

$A+B+C=0$
$2A-2B+D=2$
$4A+4B-4C=0$
$8A-8B-4D=0$
$A=\frac{1}{4}, B=-\frac{1}{4}$
$C=0, D=1$

2

1

so

$$\frac{2u^2}{u^4-16} = \frac{\frac{1}{4}}{u-2} + \frac{-\frac{1}{4}}{u+2} + \frac{1}{u^2+4}$$

$$\Rightarrow \int\frac{2u^2}{u^4-16}du = \frac{1}{4}\int\frac{1}{u-2}du - \frac{1}{4}\int\frac{1}{u+2}du + \int\frac{1}{u^2+4}du$$

$$= \frac{1}{4}\ln|u-2| - \frac{1}{4}\ln|u+2| + \int\frac{1}{u^2+2^2}$$

$$= \frac{1}{4}\ln|u-2| - \frac{1}{4}\ln|u+2| + \frac{1}{2}\tan^{-1}\left(\frac{u}{2}\right) + C$$

$$= \frac{1}{4}\ln|\sqrt{x}-2| - \frac{1}{4}\ln|\sqrt{x}+2| + \frac{1}{2}\tan^{-1}\left(\frac{\sqrt{x}}{2}\right) + C$$

3

3

Q#4[6 Marks] Evaluate $\int (\tan x)^3 (\sec x)^{\frac{3}{2}}\, dx$ $= \int \tan x \sec x \tan^2 x (\sec x)^{\frac{1}{2}} dx$

$= \int \tan x \cdot \sec x (\sec^2 x - 1)(\sec x)^{\frac{1}{2}} dx$ Put $u = \sec x$, $du = \sec x \tan x\, dx$

$= \int \tan x \cdot \sec x (u^2 - 1) u^{\frac{1}{2}} \frac{du}{\sec x \tan x}$

$= \int (u^2 - 1) u^{\frac{1}{2}} du = \int u^{\frac{5}{2}} - u^{\frac{1}{2}} du = \frac{2}{7} u^{\frac{7}{2}} - \frac{2}{3} u^{\frac{3}{2}} + C$

$= \frac{2}{7} \sec^{\frac{7}{2}} x - \frac{2}{3} \sec^{\frac{3}{2}} x\, dx$

Q#5[8 Marks] Write the partial fractional decomposition form (in simplest form) for the following integrals (don't evaluate for constant):

1. $\frac{x+1}{x^4+x} = \frac{x+1}{x(x^3+1)} = \frac{x+1}{x(x+1)(x^2-x+1)} = \frac{1}{x(x^2-x+1)} = \frac{A}{x} + \frac{Bx+C}{x^2-x+1}$

2. $\frac{3}{x^4-4x^2} = \frac{3}{x^2(x^2-4)} = \frac{3}{x^2(x-2)(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-2} + \frac{D}{x+2}$

3. $\frac{x}{(x-2)^2(x^2+64)} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{Cx+D}{x^2+64}$

4. $\frac{x^2+1}{(x+1)^2(x+4)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+4)} + \frac{D}{(x+4)^2}$

Q#6[4+2+2 Marks] Let $f(x)=\frac{1}{3}x^{\frac{3}{2}}-\sqrt{x}, 1\le x\le 16$ denote to the equation of a smooth curve $\mathbb{C}$.

1) Find the arc length function $s(x)$ of $\mathbb{C}$ by using $\left(1,\frac{-2}{3}\right)$ as a starting point.

$$s(x)=\int_1^x \sqrt{1+[f'(t)]^2}\,dt \qquad f'(x)=\frac{1}{2}x^{\frac{1}{2}}-\frac{1}{2}x^{-\frac{1}{2}}$$

$$(f'(x))^2=\frac{1}{4}x-\frac{1}{2}+\frac{1}{4}x^{-1}$$

$$=\int_1^x \sqrt{1+\frac{1}{4}t-\frac{1}{2}+\frac{1}{4t}}\,dt$$

$$=\int_1^x \sqrt{\frac{t}{4}+\frac{1}{2}+\frac{1}{4t}}\,dt=\int_1^x \sqrt{\frac{t+2t+1}{4t}}\,dt$$

$$=\int_1^x \sqrt{\frac{(t+1)^2}{4t}}\,dt=\int_1^x \frac{t+1}{2\sqrt{t}}\,dt=\int_1^x \frac{1}{2}t^{\frac{1}{2}}+\frac{1}{2}t^{-\frac{1}{2}}\,dt$$

$$=\left[\frac{1}{2}t^{\frac{3}{2}}\cdot\frac{2}{3}+\frac{1}{2}\cdot t^{\frac{1}{2}}\cdot\frac{2}{1}\right]_1^x=\left[\frac{1}{3}t^{\frac{3}{2}}+t^{\frac{1}{2}}\right]_1^x$$

$$=\frac{1}{3}(x)^{\frac{3}{2}}+\sqrt{x}-\left(\frac{1}{3}+1\right)=\frac{1}{3}(\sqrt{x})^3+\sqrt{x}-\frac{4}{3}$$

2) Use Part (1) to find the length of $\mathbb{C}$.

The solution:

$$L=\int_1^{16}\sqrt{1+(f'(t))^2}\,dt=s(16)=\frac{1}{3}(64)+2-\frac{4}{3}$$

$$=22$$

3. Let Ω be the part of the curve $\mathbb{C}$ from x=4 to x=9. Use Part (1) to find the length of Ω.

The solution:

$$L=\int_4^9\sqrt{1+(f'(t))^2}\,dt=\int_4^1\sqrt{1+(f'(t))^2}+\int_1^9\sqrt{1+(f'(t))^2}\,dt$$

$$=-\int_1^4\sqrt{1+(f'(t))^2}\,dt+\int_1^9\sqrt{1+(f'(t))^2}\,dt=s(9)-s(4)$$

$$=\left(\frac{1}{3}(3)^3+3-\frac{4}{3}\right)-\left(\frac{8}{3}+2-\frac{4}{3}\right)=\frac{19}{3}-1=\frac{16}{3}$$

Q#7[6 Marks] Evaluate $\int \frac{\cos x+\sin x}{\sin(2x)}dx$ $= \int \frac{\cos x + \sin x}{2\sin x \cos x}\,dx = \int \frac{1}{2\sin x} + \frac{1}{2\cos x}\,dx$

The solution:

$$= \frac{1}{2}\int (\csc x + \sec x)\,dx$$

$$= -\frac{1}{2}\ln|\csc x + \cot x| + \frac{1}{2}\ln|\sec x + \cot x| + C'$$

or

$$= \frac{1}{2}\ln|\csc x - \cot x| + \frac{1}{2}\ln|\sec x + \cot x| + C'$$

Q#8[6 Marks] Evaluate $\int \tan^{-1}(5x)dx$

The solution:

put $u = \tan^{-1}(5x) \Rightarrow du = \frac{5}{1+25x^2}\,dx$

$dv = dx \Rightarrow v = x$

$$\int \tan^{-1}(5x)\,dx = uv - \int v\,du = x\tan^{-1}(5x) - \int \frac{5x}{1+25x^2}\,dx$$

$$= x\tan^{-1}(5x) - \frac{1}{10}\int \frac{50x}{1+25x^2}\,dx$$

$$= x\tan^{-1}(5x) - \frac{1}{10}\ln(1+25x^2) + C'$$

Q#9[6 Marks] Find all values of p that makes $\int_2^8 \frac{1}{x^2-4p}dx$ improper (Show all your works).

The Solution

$f(x) = \frac{1}{x^2-4p}$ on $[2, 8]$. (Take p positive)

$= \frac{1}{(x-2\sqrt{p})(x+2\sqrt{p})}$

f is discts when $x = 2\sqrt{p}$ or $x = -2\sqrt{p}$.

we need $2\sqrt{p} \in [2, 8]$ or $-2\sqrt{p} \in [2, 8]$ impossible

$2 \leq 2\sqrt{p} \leq 8$

$\Rightarrow 1 \leq \sqrt{p} \leq 4 \Rightarrow 1 \leq p \leq 16$