Prince Sultan University

Deanship of Educational Services

Department of Mathematics and General Sciences

COURSE DETAILS:

Calculus II	MATH 113	FINAL EXAM
Semester:	Spring Semester --Term 172	
Date:	Tuesday May 08, 2018	
Time Allowed:	3 hours	

STUDENT DETAILS:

Student Name:	
Student ID Number:	Answer Key
Section:	
Instructor's Name:	

INSTRUCTIONS:

- You may use a scientific calculator that does not have programming or graphing capabilities. NO borrowing calculators.
- NO talking or looking around during the examination.
- NO mobile phones. If your mobile is seen or heard, your exam will be taken immediately.
- Show all your work and be organized.
- You may use the back of the pages for extra space, but be sure to indicate that on the page with the problem.

GRADING:

	Page 1	Page 2	Page 3	Page 4	Page 5	Total	Total
Questions	1,2,3	4	5,6	7,8,9	10,11	11	
Marks	16	16	15	16	17	80	40

Q#1 [1+2+3 Marks] Let $G(x) = \int_4^{x^2} f(t)dt$, $f(4) = 2$ and $f'(4) = 1$. Evaluate

(a) $G(-2)$ $= \int_4^4 f(t)\,dt = \boxed{0}$ (1)

(b) $G'(-2)$. $G'(x) = \frac{d}{dx} \int_4^{x^2} f(t)\,dt = f(x^2) \cdot 2x$ (1)

$G'(-2) = f(4) \cdot (-4) = -4\,(2) = \boxed{-8}$ (1)

(c) $G''(-2)$

$G''(x) = \frac{d}{dx}\left(2x\, f(x^2)\right) = 2f(x^2) + 4x^2 f'(x^2)$ (2)

$G''(-2) = 2f(4) + 16 f'(4) = 2(2) + 16(1) = \boxed{20}$ (1)

Q#2 [6 Marks] For each integral, evaluate it if converges, otherwise state it diverges (justify your answer).

(a) $\int_e^{\infty} \frac{dx}{x(\ln x)^2}$ $= \lim_{t\to\infty} \int_e^t \frac{dx}{x(\ln x)^2} = \lim_{t\to\infty} \left.\frac{-1}{\ln x}\right|_e^t$

$= \lim_{t\to\infty} \left[\frac{-1}{\ln t} - \frac{-1}{1}\right] = 1.$ Convergent

(1) (1)

$\int \frac{dx}{x(\ln x)^2}$. Let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$\Rightarrow \int u^{-2}\,du = \frac{-1}{u} + C$

$= -\frac{1}{\ln x} + C$ (1)

(b) $\int_1^{e} \frac{dx}{x(\ln x)^2}$ $= \lim_{t\to 1^+} \int_t^e \frac{dx}{x(\ln x)^2}$ (1) $= \lim_{t\to 1^+} \left.\frac{-1}{\ln x}\right|_t^e$

$= \lim_{t\to 1^+} \left[-1 - \frac{-1}{\ln t}\right] = -1 + \infty = \infty$ divergent.

(1) (1)

Q#3 [4 Marks] Find the value of t for which the average value of $y = (x-1)^2$ over $[1, t]$ is 3.

$f_{ave} = \frac{1}{t-1} \int_1^t (x-1)^2\,dx = 3 \Rightarrow \left.\frac{(x-1)^3}{3}\right|_1^t = 3(t-1)$ (1)

$\Rightarrow \frac{(t-1)^3}{3} = 3(t-1) \Rightarrow (t-1)^2 = 9 \Rightarrow t-1 = 3 \Rightarrow t = 4$

or $t-1 = -3$ (rejected) (1)

(1)

$\therefore \boxed{t = 4}$ (1)

Q#4 [2+3+3+4+4 Marks] Evaluate each integral.

(a) $\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta$ let $u = 1+\sec\theta \Rightarrow du = \sec\theta\tan\theta\,d\theta$

$= \int \frac{du}{u} = \ln|u| + c = \ln|1+\sec\theta| + c$ (1)

(1)

(b) $\int_0^4 |x^2-4|\,dx$ (Show your work in details)

$x^2-4=0 \Rightarrow x = \pm 2$

-2, 2: $+$ | $-$ | $+$

$= \int_0^2 (-x^2+4)\,dx + \int_2^4 (x^2-4)\,dx$ (1)

$= \left(-\frac{x^3}{3}+4x\right)_0^2 + \left(\frac{x^3}{3}-4x\right)_2^4 = \left(-\frac{8}{3}+8\right) + \left(\frac{64}{3}-16-\frac{8}{3}+8\right)$

(1)

$= \frac{16}{3} + \frac{32}{3} = \boxed{16}$ (1)

(c) $\int \cos\sqrt{t}\,dt$ Let $x = \sqrt{t} \Rightarrow x^2 = t \Rightarrow 2x\,dx = dt$

u	dv
x $+$	$\cos x$
1 $-$	$\sin x$
0	$-\cos x$

$= \int 2x\cos x\,dx = 2[x\sin x + \cos x] + C$

(1)

$= 2x\sin x + 2\cos x + C$ (2)

(d) $\int \frac{dx}{x^3-x^2}$ $\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$ (1)

$B = -1$ $(\frac{1}{2})$

$C = 1$ $(\frac{1}{2})$

$1 = Ax(x-1) + B(x-1) + Cx^2$

$x=2$: $1 = 2A + B + 4C \Rightarrow 1 = 2A - 1 + 4 \Rightarrow 2A = -2 \Rightarrow A = -1$ $(\frac{1}{2})$

$\Rightarrow \int \frac{dx}{x^3-x^2}dx = \int \left(\frac{-1}{x} - \frac{1}{x^2} + \frac{1}{x-1}\right)dx = -\ln|x| + \frac{1}{x} + \ln|x-1| + C$

(1.5)

(e) $\int \frac{dx}{e^x\sqrt{4-e^{-2x}}}$ Let $u = e^{-x} \Rightarrow du = -e^{-x}\,dx$ (1)

$= -\int \frac{du}{\sqrt{4-u^2}} = -\sin^{-1}\frac{u}{2} + C = -\sin^{-1}\left(\frac{e^{-x}}{2}\right) + C$

(1)   (1)   (1)

Q#5 [4 Marks] If f is continuous and $\int_0^2 f(x)\,dx = 8$, evaluate $\int_0^{\pi/2} f(2\sin\theta)\cos\theta\,d\theta$.

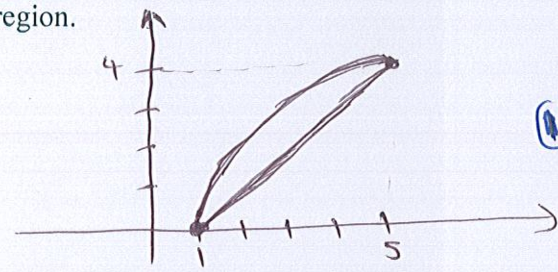
Let $u = 2\sin\theta \Rightarrow du = 2\cos\theta\, d\theta$ $\quad\theta = 0 \Rightarrow u = 0$, $\quad\theta = \frac{\pi}{2} \Rightarrow u = 2$

$$\int_0^{\frac{\pi}{2}} f(2\sin\theta)\cos\theta\, d\theta = \frac{1}{2}\int_0^2 f(u)\, du = \frac{1}{2}(8) = 4$$

(1) (1) (1) (1)

Q#6 [2+3+3+3 Marks] Consider the region bounded by the curves $y = 2\sqrt{x-1}$ and $y = x-1$.

(a) Sketch the graph of this region.



(1)

$2\sqrt{x-1} = x-1$

$4(x-1) = x^2 - 2x + 1$

$x^2 - 6x + 5 = 0$

$x = 5,\ x = 1$ (1)

(b) Calculate the area of the region. (Show your work in details)

$$A = \int_1^5 \left(2\sqrt{x-1} - (x-1)\right) dx = \left(\frac{4}{3}(x-1)^{3/2} - \frac{x^2}{2} + x\right)\Big|_1^5$$

(1) (1)

$$= \left(\frac{4}{3}4^{1.5} - \frac{25}{2} + 5\right) - \left(-\frac{1}{2} + 1\right) = \frac{19}{6} - \frac{1}{2} = \boxed{\frac{8}{3}}$$

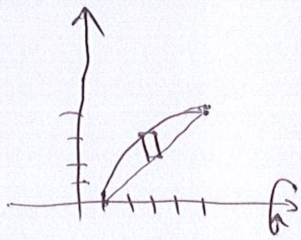
(1)

(c) Set up (do not evaluate) the integral that gives the volume when the bounded region is rotated about the x-axis.

$IR = x - 1 \ (\tfrac{1}{2}) \qquad OR = 2\sqrt{x-1} \ (\tfrac{1}{2})$

$$V = \pi \int_1^5 \left[4(x-1) - (x-1)^2\right] dx \quad (2)$$

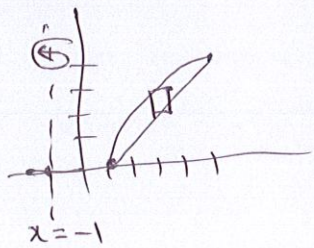
$$= \pi \int_1^5 (-x^2 + 6x - 5)\, dx$$



(d) Set up (do not evaluate) the integral that gives the volume when the bounded region is rotated about the line $x = -1$.

$$V = \int_1^5 \left[2\pi(x+1)\left[2\sqrt{x-1} - (x-1)\right]\right] dx$$

(1) (1) (1)



Q#7 [5 Marks] Find the area of the surface generated by rotating the curve $f(x)=\sqrt{3-x}, \quad 1\le x\le 3$, about the x-axis. (Show your work in details)

$$\frac{dy}{dx} = \frac{-1}{2\sqrt{3-x}} \Rightarrow 1+\left(\frac{dy}{dx}\right)^2 = 1+\frac{1}{4(3-x)} = \frac{12-4x+1}{4(3-x)} = \frac{13-4x}{4(3-x)} \quad 1$$

$$S = \int_1^3 2\pi\sqrt{3-x}\,\frac{\sqrt{13-4x}}{2\sqrt{3-x}}\,dx \quad 1 \quad = \pi\int_1^3 \sqrt{13-4x}\,dx \quad 1$$

$$= \frac{2\pi}{3}(13-4x)^{3/2}\Big|_1^3 \quad 1 \quad = 2\pi(1-27) = \boxed{\frac{-52}{3}} \quad 1$$

Q#8 [3 Marks] Determine whether the sequence $a_n=\dfrac{\tan^{-1}n}{n}$ converges or diverges. Justify your answer.

$$\lim_{n\to\infty}\frac{\tan^{-1}n}{n} \quad 1 \quad = \frac{\pi/2}{\infty} = 0 \quad 1 \quad \Rightarrow \text{seqn. converges because limit exists.} \quad 1$$

Q#9 [5+3 Marks] Determine whether each series is absolutely convergent or conditionally convergent.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+5}$

$$\sum_{n=1}^{\infty}\left|\frac{(-1)^n}{\sqrt{n}+5}\right| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+5} \qquad \text{let } a_n=\frac{1}{\sqrt{n}+5},\ b_n=\frac{1}{\sqrt{n}}$$

$\Rightarrow \displaystyle\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n}+5} = 1 > 0.$ 1 since b_n is a divergent p-series ($p=\frac{1}{2}<1$) $\Rightarrow$ series $\displaystyle\sum_{n=1}\frac{1}{\sqrt{n}+5}$ is div. 1 by L.C.T.

Now, let $b_n=\dfrac{1}{\sqrt{n}+5} \Rightarrow \displaystyle\lim_{n\to\infty} b_n = 0$ & $b_{n+1}<b_n$ since $\dfrac{1}{\sqrt{n+1}+5} < \dfrac{1}{\sqrt{n}+5}$ 1 $\Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}+5}$ is Conv. by A.S.T. 1 $\Rightarrow$

the series is conditionally convergent. 1

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{3^n}$

$\displaystyle\sum_{n=1}^{\infty}\left|\frac{\sin n}{3^n}\right|$ is convergent by C.T. since $|\sin n|<1 \Rightarrow \left|\dfrac{\sin n}{3^n}\right| < \dfrac{1}{3^n}$ and $\sum\dfrac{1}{3^n}$ is a conv. geom. ser. 1 because $r=\frac{1}{3}<1 \Rightarrow$ series $\displaystyle\sum_{n=1}\left|\frac{\sin n}{3^n}\right|$ is conv. by C.T. 1

$\Rightarrow \displaystyle\sum_{n=1}^{\infty}\frac{\sin n}{3^n}$ is absolutely convergent 1

Q#10 [4+6 Marks] Find the radius of convergence and interval of convergence of each series.

(a) $\sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!}$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{x^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{x^{n+1}}\right| = \frac{|x|}{n+2}$ 1

$\lim_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$ 1 $\Rightarrow$ series always conv. $\Rightarrow I = (-\infty, \infty)$ 1, $R = \infty$. 1

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(x+6)^n$ $\quad \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\sqrt{n+1}}{8^{n+1}}(x+6)^{n+1} \cdot \frac{8^n}{\sqrt{n}\,(x+6)^n}\right| = \frac{\sqrt{n+1}}{8\sqrt{n}}|x+6|$

$\lim_{n\to\infty} \frac{\sqrt{n+1}}{8\sqrt{n}}|x+6| = \frac{|x+6|}{8}$ 1 $\Rightarrow \frac{|x+6|}{8} < 1 \Rightarrow |x+6| < 8 \Rightarrow$

$-8 < x+6 < 8 \Rightarrow -14 < x < 2$

$x = -14$: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n}(-8)^n = \sum_{n=1}^{\infty} (-1)^n \sqrt{n}$ 1 since $\lim_{n\to\infty} (-1)^n \sqrt{n}$ D.N.E.

$\Rightarrow$ series div. by Test for div. 1

$x = 2$: $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{8^n} 8^n = \sum_{n=1}^{\infty} \sqrt{n}$ div. by T. for div. 1 since $\lim_{n\to\infty} \sqrt{n} = \infty$

$\Rightarrow I = (-14, 2)$ 1 $\quad R = 8$ 1

Q#11 [3+4 Marks] Determine whether each series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{e^{2n}}{6^{n-1}} = 6\sum_{n=1}^{\infty} \left(\frac{e^2}{6}\right)^n = e^2 + \frac{e^4}{6} + \frac{e^6}{6^2} + \frac{e^8}{6^3} + \cdots$ G.S. with

$a = e^2$ and $r = \frac{e^2}{6}$ 1 > 1 1 $\Rightarrow$ divergent. 1

(b) $\sum_{n=3}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]^2}$ $\quad f(x) = \frac{1}{x \ln x [\ln(\ln x)]^2}$ is positive Cont. for $x \geq 3$

Also $\frac{1}{(x+1)\ln(x+1)[\ln(\ln(x+1))]^2} < \frac{1}{x \ln x [\ln(\ln x)]^2}$ i.e $f(x+1) < f(x) \Rightarrow$ its

decreasing $\Rightarrow$ conditions ~~of~~ Int. Test hold. 1

$\int_3^{\infty} \frac{1}{x \ln x [\ln(\ln x)]^2} dx$ $\quad$ let $u = \ln(\ln x) \Rightarrow du = \frac{1}{\ln x} \cdot \frac{1}{x} dx$

$\int \frac{du}{u^2} = \frac{-1}{u} + C$

$\Rightarrow \lim_{t\to\infty} \frac{-1}{\ln(\ln x)}\Big|_3^t$ 1 $= \lim_{t\to\infty} \left[\frac{-1}{\ln(\ln t)} - \frac{-1}{\ln(\ln 3)}\right]$

$= \frac{1}{\ln(\ln 3)}$ 1 $\Rightarrow$ Conv. $\Rightarrow$ series converges by I. Test 1