Prince Sultan University

Math 113
Final Exam
Second Semester, Term 162
May 13, 2017

Time Allowed: 3 hours

Student Name:________________ ____________________ ________________________

Student ID #: ____________________

Serial Class #:______ Section #:

Circle your instructor's Name:

1. Dr. Jamiiru Luttamaguzi 2. Prof. Wasfi Shatanawi

Important Instructions:

1. Using notes or any textbook is not allowed during the exam.
2. Talking during the examination is NOT allowed.
3. Your exam will be taken immediately if your mobile phone is seen or heard.
4. Looking around or making an attempt to cheat will result in your exam being cancelled.
5. This examination has **6** problems. Make sure your paper has all these problems.

Problems	Max points	Student's Points
Q#1	12	
Q#2	24	
Q#3	18	
Q#4	10	
Q#5, Q#6	16	
Total	80	

Q#1 [4 Marks Each] Evaluate the following integrals:

1. $\int \frac{\sinh(x)}{e^x}\,dx$

Hint: Use the definition of sinh

The Solution:

$$\boxed{\sinh(x) = \frac{e^x - e^{-x}}{2}}$$ 1 Mark

$$\Rightarrow \frac{\sinh(x)}{e^x} = \frac{e^x - e^{-x}}{2e^x} = \boxed{\frac{1}{2} - \frac{1}{2}e^{-2x}}$$ 1 Mark

$$\Rightarrow \int \frac{\sinh(x)}{e^x}\,dx = \int \frac{1}{2}\,dx - \frac{1}{2}\int e^{-2x}\,dx$$

$$= \boxed{\frac{1}{2}x} - \boxed{\frac{1}{2}\frac{e^{-2x}}{-2}} + c$$

$$= \frac{1}{2}x + \frac{1}{4}e^{-2x} + c$$

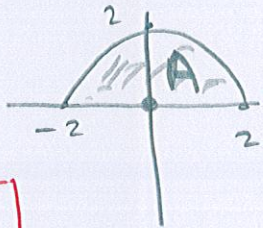
1 Mark 1 Mark

-1 Mark if c is missing

We Cancelled this question

2. $\int_{-2}^{2} \sqrt{4 - x^2}\,dx$ [By interpreting it as an area]

The Solution: put $y = \sqrt{4 - x^2} \Rightarrow y^2 + x^2 = 4$



semi cirle with center (0,0) and radius 2

3 marks

$$\int_{-2}^{2} \sqrt{4 - x^2}\,dx = \boxed{\frac{1}{2}A}$$ 1 mark

$$= \frac{1}{2}\pi(2)^2 = \boxed{2\pi}$$ 2 mark

3. $\int_{2}^{\infty} \frac{1}{(x-1)^3}\,dx$

The solution:

$$\int_{2}^{+\infty} \frac{dx}{(x-1)^3} = \boxed{\lim_{t \to +\infty} \int_{2}^{t} (x-1)^{-3}\,dx}$$ 2 Mark

$$= \boxed{\lim_{t \to +\infty} \left.\frac{(x-1)^{-2}}{-2}\right]_{2}^{t}}$$ 1 mark

$$= \boxed{\lim_{t \to +\infty} \left[\frac{-1}{2(t-1)^2} + \frac{1}{2}\right]} = \boxed{\frac{1}{2}}$$

1 mark 2 mark

Q#2[6 Marks Each] Test the following series for convergence [Justify your answer]:

1. $\sum_{n=1}^{+\infty} \frac{(-1)^n 4^{2n}}{8^{n+1}}$

The Solution:

$$= \sum_{n=1}^{+\infty} \frac{(-1)^n 16^n}{8^n \cdot 8} = \sum_{n=1}^{+\infty} \frac{1}{8}\left(\frac{-16}{8}\right)^n$$

1 Mark

$$= \sum_{n=1}^{\infty} \frac{1}{8}(-2)^n$$

1 Mark

1 Mark Geometric $r=-2$

$|r| = 2 > 1$ 1 Mark

So diverges

2 Marks

2. $\sum_{n=1}^{+\infty} \frac{\sqrt[4]{x^{12}+x+1}+x^2+3}{x^4+2x+3}$

The Solution:

$$a_n = \frac{\sqrt[4]{x^{12}+x+1}+x^2+3}{x^4+2x+3}, \quad b_n = \frac{x^3}{x^4} = \frac{1}{x}$$

1 Mark.

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = \lim_{n\to+\infty} \frac{\sqrt[4]{x^{16}+x^5+x^4}+x^3+3x}{x^4+2x+3} = \lim_{n\to+\infty} \frac{\sqrt[4]{1+\frac{x^5}{x^{16}}+\frac{1}{x^4}}+\frac{1}{x}+\frac{3}{x^2}}{1+\frac{2}{x^3}+\frac{3}{x^4}} = 1$$

2 Marks

So both series converge or both diverge.

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{x}$ diverges (p-series $p=1$),

1 Mark

We conclude by limit comparision test, $\sum a_n$ diverges

2 Marks.

3. $\sum_{n=2}^{+\infty} (-1)^n \frac{1}{n}$

The Solution:

Alternating series.

$a_n = \frac{1}{n}$

① $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$ is a decreasing sequence

② $\lim_{n\to+\infty} \frac{1}{n} = 0$. So by Alternating series test, the series converges.

4. $\sum_{n=1}^{+\infty} \left(\frac{2n+1}{n+4}\right)^{2n}$

The Solution:

$$a_n = \left(\frac{2n+1}{n+4}\right)^{2n}$$

$$\sqrt[n]{|a_n|} = \sqrt[n]{\left(\frac{2n+1}{n+4}\right)^{2n}} = \left(\frac{2n+1}{n+4}\right)^2$$

2 Marks

2 Marks $L = \lim_{n\to+\infty} \sqrt[n]{|a_n|} = \lim_{n\to+\infty} \left(\frac{2n+1}{n+4}\right)^2 = 4 > 1$

2 Marks By root test, the series diverges

Q#3 [6 Marks Each] Evaluate the following integrals:

1. $\int xe^{2x}\,dx$

put $u = x \Rightarrow du = dx$ 1 mark

$dv = e^{2x}\,dx \Rightarrow v = \int e^{2x}\,dx = \frac{1}{2}e^{2x}$ 1 Mark

$\int x e^{2x}\,dx = uv - \int v\,du = \frac{1}{2}x e^{2x} - \int \frac{1}{2} e^{2x}\,dx$

1 Mark

$= \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x} + c$

1 mark 1 mark 1 mark

2. $\int \frac{1}{\sqrt[3]{x}-\sqrt{x}}\,dx$

put $x = u^6 \Rightarrow dx = 6u^5\,du$ 1 Mark

$\int \frac{1}{\sqrt[3]{x}-\sqrt{x}}\,dx = \int \frac{6u^5}{u^2-u^3}\,du = \int \frac{6u^5}{u^2(1-u)}\,du = 6\int \frac{u^3}{1-u}\,du$

1 mark

$$\begin{array}{r|l} & -u^2-u-1 \\ \hline -u+1 & u^3 \\ & \mp u^3 \pm u^2 \\ \hline & u^2 \\ & \mp u^2 \pm u \\ \hline & u \\ & \mp u \pm 1 \\ \hline & 1 \end{array}$$

$= 6\left[\int (-u^2-u-1) + \frac{1}{1-u}\,du\right]$ 1 mark

$= 6\int(-u^2-u-1)\,du + 6\int \frac{1}{1-u}\,du$ 1 mark

$= 6\left[-\frac{u^3}{3} - \frac{u^2}{2} - u\right] - 6\ln|1-u| = -\frac{1}{2}x^{\frac{1}{2}} - \frac{1}{3}x^{\frac{1}{3}} - 6x^{\frac{1}{6}} - \ln|1-x^{\frac{1}{6}}| + c$

1 Mark 1 Mark

−1 mark if C is missing

3. $\int \sqrt{\sin x}\cos^3(x)\,dx$

$= \int \cos x \sqrt{\sin x} \cdot \cos^2 x\,dx$ 1 mark

$= \int \cos x \sqrt{\sin x} \cdot (1-\sin^2 x)\,dx$ 1 mark

put $u = \sin x$

$du = \cos x\,dx$

$dx = \frac{du}{\cos x}$

$= \int \cos x \sqrt{u}\,(1-u^2)\frac{du}{\cos x}$ 1 mark

$= \int u^{\frac{1}{2}} - u^{\frac{5}{2}}\,du$

$= \frac{2}{3}u^{\frac{3}{2}} - \frac{2}{7}u^{\frac{7}{2}} + c$ 2 marks

$= \frac{2}{3}\sin^{\frac{3}{2}} x - \frac{2}{7}\sin^{\frac{7}{2}} x + c$ 1 mark

−1 mark if C is missing

Q#4[10 Marks] Find the interval and radius of convergence of the series $\sum_{n=1}^{+\infty} \frac{(-1)^n (x+2)^n}{3^n \sqrt{n}}$

The solution: $a_n = \frac{(-1)^n (x+2)^n}{3^n \sqrt{n}}$, $|a_n| = \frac{|x+2|^n}{3^n \sqrt{n}}$

$a_{n+1} = \frac{(-1)^{n+1} (x+2)^{n+1}}{3^{n+1} \sqrt{n+1}}$, $|a_{n+1}| = \frac{|x+2|^{n+1}}{3^{n+1} \sqrt{n+1}}$

1 Mark

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+2|^{n+1}}{3^{n+1}\sqrt{n+1}} \cdot \frac{3^n \sqrt{n}}{|x+2|^n} = \frac{|x+2|}{3}\sqrt{\frac{n}{n+1}}$$

1 Mark

$$L = \lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to+\infty} \frac{|x+2|}{3}\sqrt{\frac{n}{n+1}} = \frac{|x+2|}{3}$$

The series converges if $L < 1 \Rightarrow \frac{|x+2|}{3} < 1$

2 Marks

$\Rightarrow |x+2| < 3 \Rightarrow -3 < x+2 < 3 \Rightarrow -5 < x < 1$

At $x = -5$, the series $= \sum_{n=1}^{+\infty} \frac{(-1)^n(-3)^n}{3^n \sqrt{n}} = \sum_{n=1}^{+\infty} \frac{1}{n^{\frac{1}{2}}}$ diverges p-series $p = \frac{1}{2} < 1$

2 Marks

At $x = 1$, the series $= \sum_{n=1}^{+\infty} \frac{(-1)^n (3)^n}{3^n \sqrt{n}} = \sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$

2 Marks

$a_n = \frac{1}{\sqrt{n}}$

(1) (a_n) is decreasing

(2) $\lim_{n\to+\infty} a_n = 0$

converges by alternating series test.

1 Mark

The interval of convergence $= (-5, 1]$

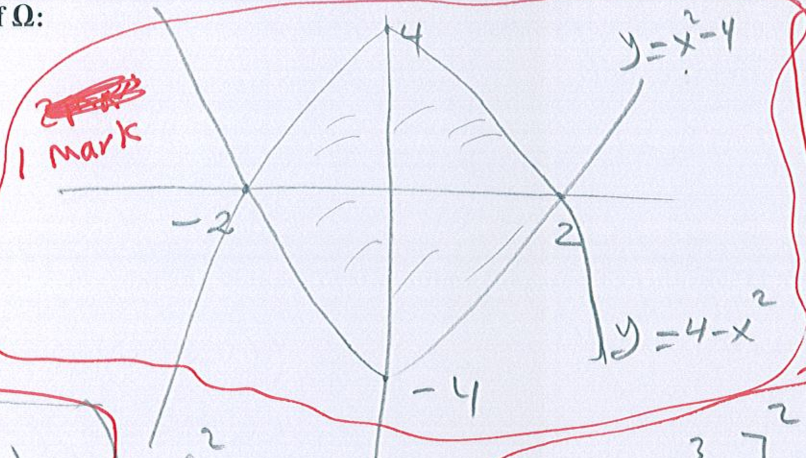
The radius of convergence $= \frac{1 - -5}{2}$

1 Mark

$= \frac{6}{2} = 3$

Q#5[2+4+4 Marks] Let Ω denoted to the region bounded by the curves $y=4-x^2$ and $y=x^2-4$.

1. **Sketch the graph of Ω:**



1 mark

$4-x^2=x^2-4$

$8=2x^2$

$x^2=4$

$x=\pm 2$

1 marks

2. **Find the area of Ω:**

$$A=\int_{-2}^{2}(4-x^2)-(x^2-4)\,dx=\int_{-2}^{2}8-2x^2\,dx=8x-\frac{2x^3}{3}\Big]_{-2}^{2}$$

2 marks 1 mark

$$=16-\frac{16}{3}-\left(-16+\frac{16}{3}\right)=32-\frac{32}{3}$$

1 mark.

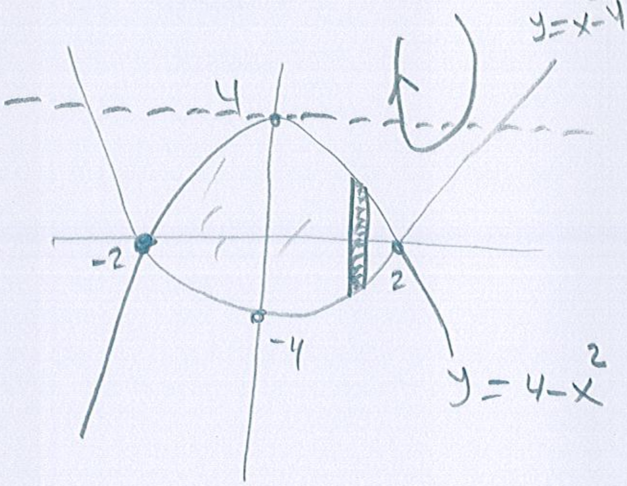
3. **Set up an integral that represent the volume of the solid generated by rotating Ω about the line $y=4$ by using discs\washers:**

$$V=\int_{-2}^{2}\pi\left(4-(x^2-4)\right)^2-\pi\left(4-(4-x^2)\right)^2\,dx$$

$$=\int_{-2}^{2}\pi\left(8-x^2\right)^2-\pi x^4\,dx$$

1 mark 1 mark.

2 marks



Q# 6[6 Marks] Set up an integral that represents the length of the arc: $y=\tan x,\quad 0\le x\le\frac{\pi}{3}$.

The solution:

$y=\tan x \Rightarrow dy=\sec^2 x\,dx$. 2 marks

$$L=\int_0^{\pi/3}\sqrt{1+(y')^2}\,dx=\int_0^{\pi/3}\sqrt{1+\sec^4 x}\,dx$$

2 marks 2 marks.