Prince Sultan University

Math 113
Final Exam
Second Semester, Term 161
January 10, 2017

Time Allowed: 3 hours

Key

Student Name: ____________ ____________ ____________

Student ID #: ____________

Serial Class #: ____ Section #:

Circle your instructor's Name:

1. Dr. Baha Abdullah 2. Dr. Jamiiru Luttamaguzi 3. Prof. Wasfi Shatanawi

Important Instructions:

1. Using notes or any textbook is not allowed during the exam.
2. Talking during the examination is NOT allowed.
3. Your exam will be taken immediately if your mobile phone is seen or heard.
4. Looking around or making an attempt to cheat will result in your exam being cancelled.
5. This examination has **8** problems. Make sure your paper has all these problems.

Problems	Max points	Student's Points
Q#1	12	
Q#2, Q#3	16	
Q#4	12	
Q#5	12	
Q#6	12	
Q#7, Q#8	16	
Total	80	

Q#1 [4 Marks Each] Evaluate the following integrals:

1. $\int_0^{\frac{\pi}{4}} \frac{\sin(2x)}{\cos x}\,dx$

The Solution:

(1)

$$\int_0^{\pi/4} \frac{\sin(2x)}{\cos x}\,dx = \int_0^{\pi/4} \frac{2\sin x\cos x}{\cos x}\,dx = 2\int_0^{\pi/4} \sin x\,dx$$

(1) (1)

$$= -2\cos x\Big]_0^{\pi/4} = -2\cos\frac{\pi}{4} + 2\cos 0$$

(1)

$$= -2\cdot\frac{1}{\sqrt{2}} + 2$$

$$= \boxed{2-\sqrt{2}}$$

2. $\int \sqrt[4]{e^{8x}}\,dx$

The Solution:

$$= \underbrace{\int (e^{8x})^{\frac{1}{4}}dx = \int}_{(2)} e^{2x}\,dx = \boxed{\underset{(1)}{\frac{1}{2}e^{2x}} + \underset{(1)}{c}}$$

(1) (1)

3. $\int_1^{\infty} \frac{1}{(x+1)^3}\,dx$

The solution:

$$\int_1^{\infty} \frac{1}{(x+1)^3}\,dx = \lim_{t\to\infty}\int_1^t (x+1)^{-3}\,dx = \lim_{t\to\infty} \frac{(x+1)^{-2}}{-2}\Big]_1^t$$

$$= \lim_{t\to\infty}\left[\frac{-1}{2(t+1)^2} + \frac{1}{8}\right] = \boxed{\frac{1}{8}}$$

(1)

(1)

Q#2[4 Marks] Test the following series for convergence and find the sum of the convergent series:

1. $\sum_{n=1}^{+\infty} \frac{(-1)^n 4^{2n}}{8^{2n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^n 16^n}{64^n \cdot 8} = \sum_{n=1}^{\infty} \frac{1}{8}\left(\frac{-16}{64}\right)^n = \sum_{n=1}^{\infty} \frac{1}{8}\left(-\frac{1}{4}\right)^n$

The Solution:

(2) Geom. Series. $r = -\frac{1}{4}$, $|r| = \frac{1}{4} < 1$, the $\boxed{\text{Series converges}}$ to

(2) $S = \frac{a}{1-r} = \frac{\frac{1}{8}\left(-\frac{1}{4}\right)}{1+\frac{1}{4}} = \frac{-1}{32} \cdot \frac{4}{5} = \boxed{\frac{-1}{40}}$

2. $\sum_{n=1}^{+\infty} \cos\left(\frac{1}{n}\right)$

The Solution: (2) $\lim_{n \to \infty} \cos\left(\frac{1}{n}\right) = 1 \neq 1$. By divergent theorem, we conclude that the series (2) $\boxed{\text{diverges}}$.

3. $\sum_{n=2}^{+\infty} e^{-n} - e^{-(n+1)}$ $\quad a_n = e^{-n} - e^{-(n+1)}$, $n = 2, 3, 4, \dots,$

The Solution: Telescoping series.

$S_1 = a_2 = e^{-2} - e^{-3}$ (2)

$S_2 = a_2 + a_3 = (e^{-2} - \cancel{e^{-3}}) + (\cancel{e^{-3}} - e^{-4}) = e^{-2} - e^{-4}$

$S_3 = a_2 + a_3 + a_4 = (e^{-2} - \cancel{e^{-3}}) + (\cancel{e^{-3}} - \cancel{e^{-4}}) + (\cancel{e^{-4}} - e^{-5}) = e^{-2} - e^{-5}$

So $S_n = e^{-2} - e^{-(n+2)}$ (1)

$\therefore \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(e^{-2} - e^{-(n+2)}\right) = e^{-2}$.

So the series $\boxed{\text{converges}}$ to $\boxed{e^{-2}}$. (1)

Q#3 [4 Marks] Evaluate $\int (x+1)e^x \, dx$

Let $u = x+1 \Rightarrow du = dx$ (1)

$dv = e^x dv \Rightarrow v = e^x$

$\int (x+1)e^x dx = uv - \int v \, du$

$= (x+1)e^x - \int e^x dx$ (1)

$= (x+1)e^x - e^x + c$

$= \boxed{xe^x + c}$

(1) (1)

Q#4[4 Marks Each] Test the following series for convergence or divergence:

1. $\sum_{n=2}^{+\infty} \frac{(-1)^n}{\ln(n)}$ Use $\boxed{\text{Alternating series test}}$, $a_n = \frac{1}{\ln(n)}$

The solution:

① is $\{a_n\}$ decreasing.

Let $f(x) = \frac{1}{\ln x}$, $\Rightarrow f'(x) = -\frac{1}{x(\ln x)^2}$. The critical pts are $x = 0, 1$

$f'(x) < 0$ on $[2, +\infty)$

$\therefore$ $\{a_n\}$ decreases for $n \geq 2$

② $\lim_{n\to\infty} \frac{1}{\ln(n)} = 0$.

So this series satisfies the Alternating series test and hence the series $\boxed{\text{converges}}$.

2. $\sum_{n=1}^{+\infty} \frac{\sqrt[3]{n^9+4n+2}}{n^4+2n^3-1}$ $a_n = \frac{\sqrt[3]{n^9+4n+2}}{n^4+2n^3-1}$, $b_n = \frac{n^3}{n^4} = \frac{1}{n}$.

The solution:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt[3]{n^9+4n+2}}{n^4+2n^3-1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{\sqrt[3]{n^{12}+4n^4+2n^3}}{n^4+2n^3-1}$$

$$= \lim_{n\to+\infty} \sqrt[3]{\frac{n^{12}+4n^4+2n^3}{(n^4+2n^3-1)^3}} = \sqrt[3]{1} = 1 \neq 0$$

So by $\boxed{\text{limit comparison test}}$ both series converge or both diverge.

Since $\sum b_n = \sum \frac{1}{n}$ diverges ($p=1$ Harmonic) $\Rightarrow \sum \frac{\sqrt[3]{n^9+4n+2}}{n^4+2n^3-1}$ $\boxed{\text{diverges}}$.

$\sum_{n=1}^{+\infty} \left(\frac{2n+1}{n+4}\right)^{2n}$ $a_n = \left(\frac{2n+1}{n+4}\right)^{2n}$ $|a_n|^{1/n} = \left(\frac{2n+1}{n+4}\right)^2$

The Solution:

$$R = \lim_{n\to+\infty} |a_n|^{1/n} = \lim_{n\to+\infty} \left(\frac{2n+1}{n+4}\right)^2 = 4 > 1$$

By the $\boxed{\text{root test}}$, the series $\boxed{\text{diverges}}$.

Q#5[12 Marks] Find the interval and radius of convergence of the series $\sum_{n=1}^{+\infty} \frac{(-1)^n (x-3)^n}{2^n \sqrt{n}}$ $\qquad a_n = \frac{(-1)^n (x-3)^n}{2^n \sqrt{n}}$

The solution:

$\underline{x=3}$: center : series converges.

$\underline{x \neq 3}$: Use Ratio Test $|\cdot| = 1$.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^{n+1}}(x-3)^{n+1}}{2^{n+1}\sqrt{n+1}} \cdot \frac{2^n \sqrt{n}}{\cancel{(-1)^n}(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \frac{1}{2}\sqrt{\frac{n}{n+1}} \cdot |x-3| = \frac{|x-3|}{2} < 1$$

for convergence.

$\Rightarrow |x-3| < 2$. for convergence

$\boxed{\text{Radius of convergence is } 2}$

$-2 < x-3 < 2 \Rightarrow +1 < x < 5$

$\underline{x=+1}$: $\sum_{n=1}^{\infty} \frac{(-1)^n (+1-3)^n}{2^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{\cancel{(-1)^n}\cancel{(-2)^n}}{\cancel{2^n}\sqrt{n}}$

$= \sum \frac{1}{\sqrt{n}}$ diverges $p = \frac{1}{2}$

$\underline{x=5}$: $\sum_{n=1}^{\infty} \frac{(-1)^n (5-3)^n}{2^n \sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{\cancel{2^n}\sqrt{n}}$

$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by Alternating Series Test.

$\lim \frac{1}{\sqrt{n}} = 0$, $\frac{1}{\sqrt{n}}$ decreases.

$\boxed{\therefore \text{ Interval of convergence is } (1, 5]}$

Q#6 [6 Marks Each] Solve the following integrals:

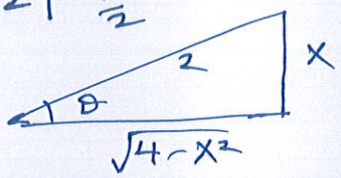
1. $\int_0^2 \sqrt{4-x^2}\,dx$

The solution:

$x = 2\sin\theta,\quad \frac{x}{2} = \sin\theta,$

$dx = 2\cos\theta\, d\theta$

x	θ
0	0
2	$\frac{\pi}{2}$



$$\int_0^2 \sqrt{4-x^2}\,dx = \int_0^{\pi} \sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta = 4\left[\left(\frac{\cos\theta\sin\theta}{2}\right)_0^{\frac{\pi}{2}} + \frac{1}{2}\int_0^{\pi/2} d\theta\right]$$

$$\cancel{= 2\left[(-1)(0) - (1)(0)\right] + 2(\pi)}$$

$$= 2\left[\cos\frac{\pi}{2}\sin\frac{\pi}{2} - \cos 0 \sin 0 + \frac{\pi}{2}\right]$$

$$= \boxed{\pi}$$

2. $\int \frac{1}{(x-2)(x^2-1)}\,dx$

The solution:

$$\frac{1}{(x-2)(x^2-1)} = \frac{1}{(x-2)(x-1)(x+1)} = \frac{A}{x-2} + \frac{B}{x-1} + \frac{C}{x+1}$$

$$= \frac{A(x+1)(x-1) + B(x-2)(x+1) + C(x-2)(x-1)}{(x-2)(x-1)(x+1)}$$

$1 = A(x+1)(x-1) + B(x-2)(x+1) + C(x-2)(x-1)$

$x = 1 \Rightarrow 1 = -2B \Rightarrow B = -1/2$

$x = -1 \Rightarrow 1 = 6C \Rightarrow C = 1/6$

$x = 2 \Rightarrow 1 = 3A \Rightarrow A = 1/3$.

So $\frac{1}{(x-2)(x^2-1)} = \frac{1/3}{x-2} - \frac{1/2}{x-1} + \frac{1/6}{x+1}$

$$\int \frac{1}{(x-2)(x^2-1)}\,dx = \frac{1}{3}\int\frac{dx}{x-2} - \frac{1}{2}\int\frac{dx}{x-1} + \frac{1}{6}\int\frac{dx}{x+1}$$

$$= \boxed{\frac{1}{3}\ln|x-2| - \frac{1}{2}\ln|x-1| + \frac{1}{6}\ln|x+1| + C}$$

Q#7[2+4+4 Marks] **Graph the region bounded by the curves** $y=4-x^2$, $y=3x$ **and** $0\le x\le 1$. **Then set up the integral** to find the volume of the solid generated by rotating the given region about x-axis **[Using the Disk/Washer method** and Cylindrical shells]. Do not evaluate for the integral.

The Solution:

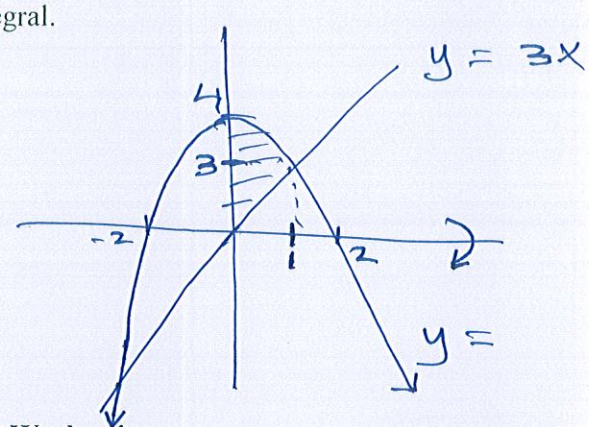
1. **The graph of the given region is:**

Intersection $4-x^2=3x$

$x^2+3x-4=0$

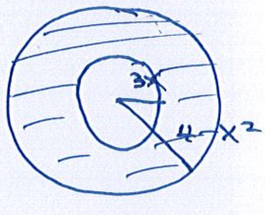
$(x+4)(x-1)=0$

$x=-4,\ 1$



2. **The integral to find the volume by using Disk and Washer is:**

$$V=\int_0^1 \pi(4-x^2)^2-\pi(3x)^2\,dx$$

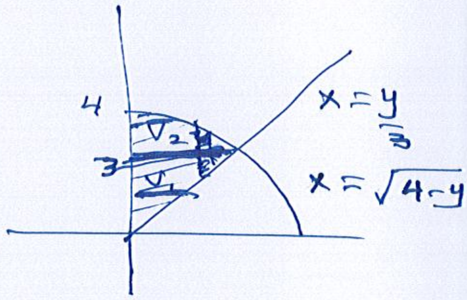


3. **The integral to find the volume by using cylindrical shells is:** $V=\int_0^1 A(x)\,dx$

$V=V_1+V_2$

$$=\int_0^3 2\pi y\cdot\frac{y}{3}\,dy+\int_3^4 2\pi y\sqrt{4-y}\,dy$$

$$=\int_0^3 \frac{2\pi y^2}{2}\,dy+\int_3^4 2\pi y\sqrt{4-y}\,dy$$



Q# 8[6 Marks] Set up an integral for the area of the surface obtained by rotating the curve $y=\tan x$, $0\le x\le\frac{\pi}{3}$ about the x-axis.

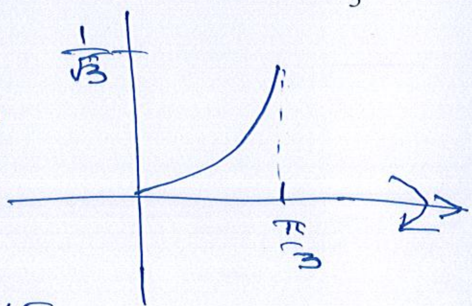
The solution: Method 1

$$S=\int_0^{\pi/3} 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_0^{\pi/3} 2\pi\tan x\sqrt{1+\sec^3 x}\,dx$$

$x=\tan^{-1}y$

$\frac{dx}{dy}=\frac{1}{1+y^2}$



$$S=\int_0^{1/\sqrt{3}} 2\pi y\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

$$=\int_0^{1/\sqrt{3}} 2\pi y\sqrt{1+\frac{1}{(1+y^2)^2}}\,dy$$